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Approximate Solution of Strongly Nonlinear Vibrations which Vary with Time

Pinakee Dey [1], Md Asaduzzaman [2], Razia Pervin [3] and M. A. Sattar [4]

[1,3,4,] Department of Mathematics,
Mawlana Bhashani Science and Technology University,
Santosh, Tangail-1902, BANGLADESH.
Ranada Prasad Shaha University, Narayangang, BANGLADESH.
email: pinakee_68@yahoo.com

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ABSTRACT

In this article an analytical technique is presented to determine approximate solutions of nonlinear oscillatory systems with slowly varying coefficients based on the combined work of extended Krylov-Bogoliubov-Mitropolskii method and harmonic balance (HB) method. The determination of the solution is systematic and easier than the existing procedures developed by several authors. The method is illustrated by an example.

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1. INTRODUCTION

The most common methods for constructing the analytical approximate solutions to the nonlinear oscillator equations are the perturbation methods. Another important method is harmonic balance (HB) method[1-2], which covers strongly nonlinear problems. One widely spread method in perturbation theory is the asymptotic method of Krylov-Bogoliubov-Mitropolskii (KBM)[3-5]. Krylov and Bogoliubov[3] originally developed this perturbation method to obtain an approximate solution of a second order nonlinear differential system which was amplified and justified by Bogoliubov and Mitropolskii[4]. Asymptotic methods for second order differential systems with slowly varying coefficients were also developed by Mitropolskii[5], Popov[6] extended this method to a damped oscillation. Murty[7] has presented a unified KBM method for both under-damped and over-damped system with constant coefficients. Shamsul[8] has presented a unified KBM method for solving an *n*-th order differential equation (autonomous) characterized by oscillatory, damped oscillatory and non-oscillatory processes with slowly varying coefficients. Recently, Roy and Shamsul[8] found an asymptotic solution of a differential system in which the coefficient changes in an exponential order of slowly varying time. In this article, we have used combining the methods of extended KBM[3-5] and Harmonic balance (HB) method[1-2] for strong nonlinearity to find the solution of nonlinear differential systems in which the coefficients change slowly and periodically with time.

2. METHOD

Let us consider the nonlinear differential system

$$\ddot{x} + (c_1 + c_2 \cos\tau + c_3^2 \sin 2\tau)x = -\varepsilon f(x,\tau), \qquad \tau = \varepsilon t \tag{1}$$

where the over-dots denote differentiation with respect to t, ε is a small parameter, c_1, c_2 and c_3 are constants, $c_2 = c_3 = O(\varepsilon)$, f is a given nonlinear function. Setting $\omega^2(\tau) = (c_1 + c_2 \cos\tau + c_3^2 \sin 2\tau)$, $\omega(\tau)$ is known as frequency.

Putting $\varepsilon = 0$ and $\tau = \tau_0$ = constant, in Eq.(1), we obtain the unperturbed solution of (1) in the form

$$x(t,0) = x_{1,0} e^{\lambda_1(\tau_0)t} + x_{-1,0} e^{\lambda_2(\tau_0)t}, \tag{2}$$

When $\varepsilon \neq 0$, we seek a solution of Eq. (1) in the form

$$x(t,\varepsilon) = x_1(t,\tau) + x_{-1}(t,\tau) + \varepsilon u_1(x_1, x_{-1}, t, \tau) + \varepsilon^2 u_2(x_1, x_{-1}, t, \tau) + \ldots, \tag{3}$$

where x_1 and x_{-1} satisfy the equations

$$\begin{aligned} \dot{x}_1 &= \lambda_1(\tau)x_1 + \varepsilon X_1(x_1, x_{-1}, \tau) + \varepsilon^2 X_1(x_1, x_{-1}, \tau)\ldots, \\ \dot{x}_{-1} &= \lambda_2(\tau)x_{-1} + \varepsilon X_{-1}(x_1, x_{-1}, \tau) + \varepsilon^2 X_{-1}(x_1, x_{-1}, \tau)\ldots, \end{aligned} \tag{4}$$

Differentiating $x(t,\varepsilon)$ two times with respect to t, substituting for the derivatives $\ddot{x}$ and x in the original equation (1) and equating the coefficient of ε, we obtain

$$\begin{aligned} &(\lambda_1 x_1 \Omega x_1 + \lambda_2 x_{-1} \Omega x_{-1}) X_1 + (\lambda_1 x_1 \Omega x_1 + \lambda_2 x_{-1} \Omega x_{-1}) X_{-1} + \lambda_1' x_1 + \lambda_2' x_{-1} - \lambda_2 X_1 - \lambda_1 X_{-1} + \\ &+ (\lambda_1 x_1 \Omega x_1 + \lambda_2 x_{-1} \Omega x_{-1} - \lambda_1)(\lambda_1 x_1 \Omega x_1 + \lambda_2 x_{-1} \Omega x_{-1} - \lambda_2) u_1 \\ &= -f^{(0)}(x_1, x_{-1}, \tau), \end{aligned} \tag{5}$$

where $\lambda_1' = \dfrac{d\lambda_1}{d\tau}$, $\lambda_2' = \dfrac{d\lambda_2}{d\tau}$, $\Omega x_1 = \dfrac{\partial}{\partial x_1}$, $\Omega x_{-1} = \dfrac{\partial}{\partial x_{-1}}$, $f^{(0)} = f(x_0, \dot{x}_0, \tau)$

Herein it is assumed that $f^{(0)}$ can be expanded in Taylor's series

$$f^{(0)} = \sum_{r_1, r_2 = 0}^{\infty} F_{r_1, r_2}(\tau) x_1^{r_1} x_{-1}^{r_2} \tag{6}$$

To obtain a solution of (1), we impose a restriction that $u_1 \cdots$ exclude the terms $x_1^{i_1} x_{-1}^{i_2}, i_1 - i_2 = \pm 1 \quad i_1, i_2 = 0, 1, 2 \cdots$. The assumption assures that $u_1 \cdots$ are free from secular type terms $te^{-\lambda_1 t}$. This restriction guarantees that the solution always excludes *secular*-type terms or the first harmonic terms, otherwise a sizeable error would occur[8]. We shall be able to transform (3) to the exact form of the KBM[3-5] solution by substituting $x_1 = a\, e^{i\varphi}/2$ and $x_{-1} = a\, e^{-i\varphi}/2$. Herein, a and φ are respectively amplitude and phase variables (see Shamsul[8]). Under this assumption, we shall able to find the unknown functions u_1 and X_1, X_{-1}.

3. EXAMPLE

As example of the above procedure, let us consider a nonlinear vibrating systems with slowly varying coefficients

$$\ddot{x} + (c_1^2 + c_2 \cos\tau + c_3^2 \sin 2\tau)x = -\varepsilon x^3, \tag{7}$$

Here over dots denote differentiation with respect to t. $x_0 = x_1 + x_{-1}$ and the function $f^{(0)}$ becomes,

$$f^{(0)} = -(x_1^3 + 3x_1^2 x_{-1} + 3x_1 x_{-1}^2 + x_{-1}^3)\,. \tag{8}$$

Following the assumption (discussed in section 2) excludes we substitute in (5) and separate it into two parts as

$$\left(\lambda_1 x_1 \partial x_1 + \lambda_2 x_{-1} \partial x_{-1}\right) X_1 + \left(\lambda_1 x_1 \partial x_1 + \lambda_2 x_{-1} \partial x_{-1}\right) X_{-1} + \lambda_1' x_1 + \lambda_2' x_{-1} - \lambda_2 X_1 - \lambda_1 X_{-1}$$
$$= -(3x_1^2 x_{-1} + 3x_1 x_{-1}^2) \tag{9}$$

and $\left(\lambda_1 x_1 \partial x_1 + \lambda_2 x_{-1} \partial x_{-1} - \lambda_1\right)\left(\lambda_1 x_1 \partial x_1 + \lambda_2 x_{-1} \partial x_{-1} - \lambda_2\right) u_1 = -(x_1^3 + x_{-1}^3)$ (10)

The particular solution of (10) is

$$u_1 = -x_1^3 / 2\lambda_1(3\lambda_1 - \lambda_2) - x_{-1}^3 / 2\lambda_2(3\lambda_2 - \lambda_1) \tag{11}$$

The particular solutions of (9) is

$$X_1 = -\lambda_1' x_1 / (\lambda_1 - \lambda_2) - 3x_1^2 x_{-1} / 2\lambda_1$$

and

$$X_{-1} = \lambda_2' x_{-1} / (\lambda_1 - \lambda_2) - 3x_1 x_{-1}^2 / 2\lambda_2 \tag{12}$$

Substituting the functional values of X_1 and X_{-1} (12) into (4) and rearranging, we obtain

$$\dot{x}_1 = \lambda_1 x_1 + \varepsilon\left(-\lambda_1' x_1 / (\lambda_1 - \lambda_2) - 3x_1^2 x_{-1} / 2\lambda_1\right)$$

and

$$\dot{x}_{-1} = \lambda_2 x_{-1} + \varepsilon\left(\lambda_2' x_{-1} / (\lambda_1 - \lambda_2) - 3x_1 x_{-1}^2 / 2\lambda_2\right) \tag{13}$$

The variational equations of ° and ¤ in the real form (° and ¨ are know as amplitude and phase) which transform (13) to

$$\dot{°} = -£°µ'/2µ \quad \text{and} \quad \dot{¤} = µ + 3£°^2/8µ \qquad (14)$$

where $µ^2 = (c_1 + c_2 \cos² + c_3^2 \sin 2²)$

The variational equation (14) is in the form of the KBM solution**.** The variational equations for amplitude and phase are usually appeared in a set of first order differential equations and solved by the numerical technique (see Shamsul[8]).

Therefore, the improved solution of the equation (7) is

$$x(t, £) = ° \cos ¨ + £ \; u_1 + \cdots, \qquad (15)$$

where $¨ = µt + ¤$ (see **Appendix A**) and °, ¤ are the solutions of the equations (14) and u_1 is given by Eq. (11).

4. RESULTS AND DISCUSSIONS

In this article an analytic technique has been presented to obtain the first order analytical approximate solutions of nonlinear differential systems with constant and varying coefficients based on the extended KBM method (by Popov) and HB method. Theoretically, the solution can be obtained up to the accuracy of any order of approximation. However owing to the rapidly growing algebraic complexity for the derivation of the function, the solution is in general confined to a low order, usually the first. In order to test the accuracy of an approximate solution obtained by a certain perturbation method, one can easily compare the approximate solution to the numerical solution (considered to be exact). Due to such a comparison concerning the of this paper, we refer to the works of Murty[7], and Shamsul[8] have been compared to the corresponding numerical solution. In this paper we have also compared the perturbation solutions (16) and of *Duffing's* equation (7) to those obtained by Range-kutta (Fourth-order) procedure.

First of all, x is calculated by (18) with initial conditions $[x(0) = 1.0000, \; \dot{x}(0) = 0.0000]$ or $[° = 1.0000, \; ¨ = .0010375]$ for $£ = .1$, $µ^2 = µ_0(c_1 + c_2 \cos² + c_3^2 \sin 2²)$. Then corresponding numerical solutions is also computed by Runge-Kutta fourth-order method. The result is shown in Fig.1(a). Also we plot existing perturbation solution in Fig.1(b) with initial conditions $[x(0) = 1.0000, \; \dot{x}(0) = 0.0000]$ or $[° = 1.0000, \; ¨ = -.000237]$ for $£ = .1$, $µ^2 = µ_0(c_1 + c_2 \cos² + c_3^2 \sin 2²)$. We see that in Fig. 1(a) the perturbation solution nicely agree with the numerical solution, but in this situation existing perturbation solution (unified method) in Fig.1(b) does not give satisfactory result. The corresponding numerical solutions have also been computed by Runge-Kutta fourth-order method. From Fig. 2(a), Fig. 3(a), we observe that the approximate solutions agree with numerical results nicely but in Fig. 2(b), Fig. 3(b) do not agree and the solution fails to give desired result.

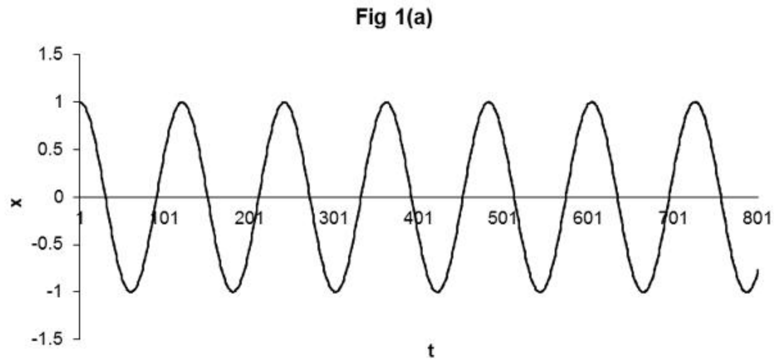

Fig 1(a): Perturbation solution (dotted line) with corresponding numerical solution (solid line) are plotted with initial conditions $^{\circ} = 1.00000,\ ^{\cdot\cdot} = -.0010375\quad [x(0) = 1.00000,\ \dot{x}(0) = 0.00000]$ **for** $e = .1,\ h = .05.$

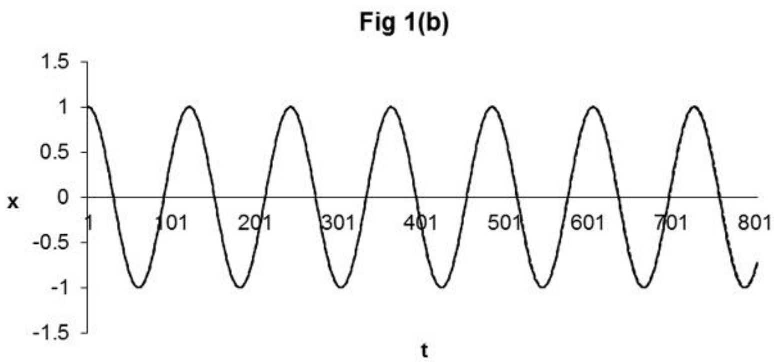

Fig 1(b): Perturbation solution (dotted line) with corresponding numerical solution (solid line) are plotted with initial conditions $^{\circ} = 1.00000,\ ^{\cdot\cdot} = -.000237\quad [x(0) = 1.00000,\ \dot{x}(0) = 0.00000]$ **for** $e = .1,\ h = .05.$

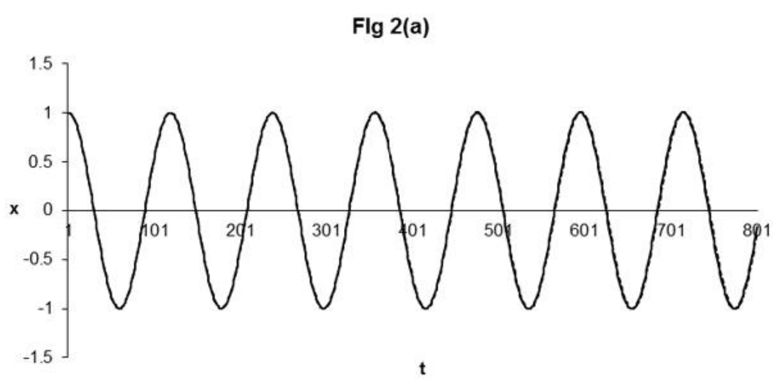

Fig 2(a): Perturbation solution (dotted line) with corresponding numerical solution (solid line) are plotted with initial conditions $^{\circ} = 1.00000,\ ^{\cdot\cdot} = -.001075[x(0) = 1.00000, \dot{x}(0) = 0.00000]$ **for** $e = .2,\ h = .05$

Fig 2(b)

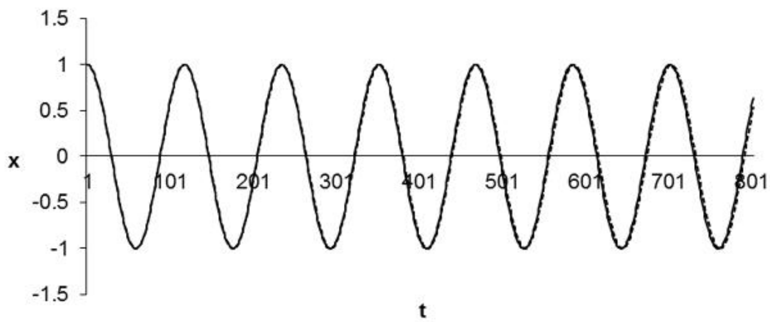

Fig 2(b): Perturbation solution (dotted line) with corresponding numerical solution (solid line) are plotted with initial conditions $^{\circ} = 1.00000,\ ^{\cdot\cdot} = -.000459\ \ [x(0) = 1.00000,\ \dot{x}(0) = 0.00000]$ **for** $e = .2,\ h = .05.$

Fig 3(a)

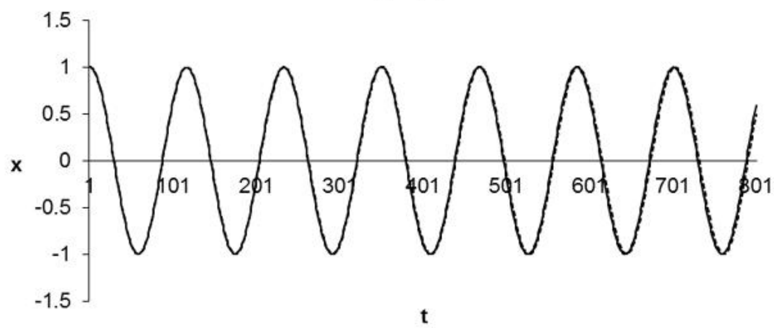

Fig 3(a): Perturbation solution (dotted line) with corresponding numerical solution (solid line) are plotted with initial conditions $^{\circ} = 1.00000,\ ^{\cdot\cdot} = -.001112\ \ [x(0) = 1.00000,\ \dot{x}(0) = 0.00000]$ **for** $e = .3,\ h = .05.$

Fig 3(b)

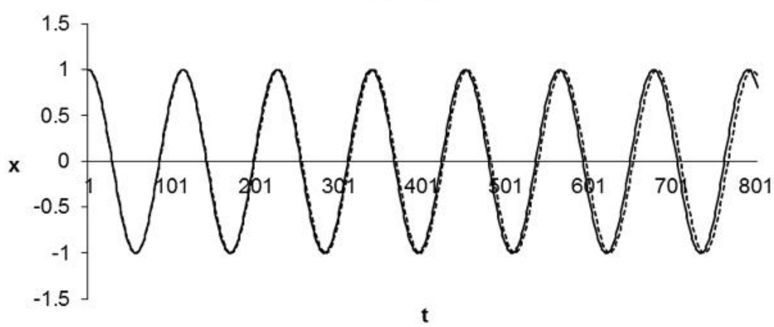

Fig 3(b): Perturbation solution (dotted line) with corresponding numerical solution (solid line) are plotted with initial conditions $^{\circ} = 1.00000,\ ^{\cdot\cdot} = -.000665\ [x(0) = 1.00000,\ \dot{x}(0) = 0.00000]$ **for** $e = .3,\ h = .05.$

5. CONCLUSION

An approximate solution of nonlinear deferential system with slowly varying coefficients has been found. This improved method gives better results than existent KBM method. The solution for different initial condition shows good coincidence with corresponding numerical solution.

6. APPENDIX -A

Let us consider nonlinear differential equation

$$\ddot{x} + (c_1^2 + c_2 \cos\theta + c_3^2 \sin 2\theta)x = -\varepsilon x^3, \qquad \text{(A.1)}$$

According to [1-2], there exits a periodic solution of **harmonic balance** method of Eq.(A.1) in the form,

$$x(t,0) = a\cos\omega + a^3 c_3 \cos 3\omega + ... \qquad \text{(A.2)}$$

where a and c_3 are constants.

By substituting Eq.(A.2) in to Eq. (A.1) and equating the coefficient of $\cos\omega$, we obtain

$$-\omega^2 + \mu^2 = \frac{3\varepsilon a^2}{4}(1 + a^2 c_3) \qquad \text{(A.3)}$$

Here $c_3 = 0$ then the Eq.(A.3), become

$$-\omega^2 + \mu^2 = \frac{3\varepsilon a^2}{4} \qquad \text{(A.4)}$$

Simplifying this equation (A.4), we get

$$\omega = 4\mu\left(\sqrt{(1 - 3\varepsilon a^2 / 4\mu^2}\right) \qquad \text{(A.5)}$$

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